COMPLETE MODULI FOR FAMILIES OVER SEMISTABLE CURVES

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This note is a research announcement, summarizing and explaining results proven and detailed in forthcoming papers [ℵ-V1], [ℵ-V2], [ℵ-C-J-V].

1. MODULI AND STACKS

1.1. **Stacks as moduli objects.** In the last two decades, it has been observed that typically a "nice" moduli problem corresponds to a Deligne-Mumford algebraic stack admitting a projective coarse moduli scheme. There are numerous examples of this phenomenon. Let us mention just a few:

1. $\overline{\mathcal{M}}_{g,n}$: the moduli of stable n-pointed curves of genus g [D-M]; which, away from small characteristics generalizes to
2. $\overline{\mathcal{M}}_{g,n}(X,d)$: Kontsevich's moduli of stable n-pointed maps of genus g and degree d; and
3. $\mathcal{B}G$: the moduli of principal homogeneous G-spaces, for a finite group G.

Deligne-Mumford stacks form a 2-category which is an extension of the category of schemes in a natural way. There is an extensive theory of cohomology, intersection theory, vector bundles, and K-theory of stacks, which was developed largely due to the importance of stacks in moduli theory. A closely related notion of $\mathbb{Q}$-varieties has been studied extensively. Some natural diophantine equations related to stacks were studied in [D-G].

1.2. **Stacks are basic objects.** In this note we hope to convince the reader that Deligne-Mumford stacks should be considered as basic objects of algebraic geometry, like schemes, and not just as objects dedicated to moduli problems. We argue as follows: a natural moduli problem of certain stable families over nodal curves is introduced; this moduli problem is not complete; a natural compactification of this moduli problem involves families over *curves with Deligne-Mumford stack structure*; the resulting complete moduli problem is a nice one, namely it is a complete Deligne-Mumford stack admitting a projective coarse moduli scheme.

Let us introduce such moduli problems.

1.3. **The problem of moduli of families.** Consider a Deligne-Mumford stack $\mathcal{M}$ admitting a projective coarse moduli scheme $\mathbf{M} \subset \mathbb{P}^N$. Given a curve C, it is often natural to consider morphisms $f : C \to \mathcal{M}$ (or equivalently, objects $f \in \mathcal{M}(C)$): in case $\mathcal{M}$ is the moduli of geometric objects, these morphisms correspond to families over C. For example, if $\mathcal{M} = \overline{\mathcal{M}}_\gamma$, then morphism $f : C \to \mathcal{M}$ correspond to families of stable curves of genus γ over C; and if $\mathcal{M} = \mathcal{B}G$ we get principal G-bundles over C. It should be obvious that it is interesting to study moduli of such objects; moreover, it is natural to study such moduli as C varies, and find a natural compactification for such moduli.

1.4. **Stable maps.** Denote by g the genus of C. In case $\mathcal{M}$ is represented by a projective scheme $X \subset \mathbb{P}^N$, a natural answer to these questions is given by the Kontsevich stacks of stable maps $\overline{\mathcal{M}}_g(X,d)$. It is tempting to mimic this construction in the case of an arbitrary stack as follows: let C be a nodal projective connected curve; then a morphism $C \to \mathcal{M}$ is said to be a *stable map* of degree d if the associated morphism to the coarse moduli scheme $C \to \mathbf{M}$ is a stable map of degree d.

It follows from our results below that this moduli problem is a Deligne-Mumford stack. A somewhat surprising point is, that it is not complete.

To see this, we fix $g = 2$ and consider the specific case of $\mathcal{B}G$ with $G = (\mathbb{Z}/3\mathbb{Z})^4$. Any smooth curve of genus 2 admits a connected principal G bundle, corresponding of a surjection $H_1(C,\mathbb{Z}) \to G$. If we let C degenerate to a nodal curve C_0 of geometric genus 1, then $H_1(C_0,\mathbb{Z}) \simeq \mathbb{Z}^3$, and since there is no surjection $\mathbb{Z}^3 \to G$, there is no connected principal G-bundle over C_0. This means that there is no limiting stable map $C_0 \to \mathcal{B}G$.

1.5. **Enter orbispace-curves.** Our main goal here is to correct this deficiency. In order to do so, we will enlarge the category of stable maps into $\mathcal{M}$. The source curve $\mathcal{C}$ of a new stable map $\mathcal{C} \to \mathcal{M}$ will acquire an orbispace structure at its nodes. Specifically, we endow it with the structure of a Deligne-Mumford stack.

It is not hard to see how these orbispace structure come about. Let S be the spectrum of a discrete valuation ring R of pure characteristic 0, with quotient field K, and let $C_K \to \eta \in S$ be a nodal curve over the generic point, together with a map $C_K \to \mathcal{M}$ of degree d, whose associated map $C_K \to \mathbf{M}$ is stable. We can exploit the fact that $\overline{\mathcal{M}}_{g,0}(\mathbf{M}, d)$ is complete; after a ramified base change on S the induced map $C_K \to \mathbf{M}$ will extend to a stable map $C \to \mathbf{M}$ ove S. Let C_{sm} be the smooth locus of the morphism $C \to S$; Abhyankar's lemma, plus a fundamental purity lemma (see 1.5.1 below) shows that after a suitable base change we can extend the map $C_K \to \mathcal{M}$ to a map $C_{\mathrm{sm}} \to \mathcal{M}$; in fact the purity lemma fails to apply only at the "new" nodes of the central fiber, namely those which are not in the closure of nodes in the generic fiber. On the other hand, if $p \in C$ is such a node, then on an étale neighborhood U of p, the curve C looks like

$$uv = t^r,$$

where t is the parameter on the base. By taking r-th roots:

$$u = u_1^r; \, v = v_1^r$$

we have a *nonsingular* cover $V_0 \to U$ where V_0 is defined by $u_1 v_1 = t$. The purity lemma applies to V_0, so the composition $V_{0K} \to C_K \to \mathcal{M}$ extends over all of V_0. There is a minimal intermediate cover $V_0 \to V \to U$ such that the family extends already over V; this V will be of the form $xy = t^{r/m}$, and the map $V \to U$ is given by $u = x^m$, $v = y^m$. Furthermore, there is an action of the group $\boldsymbol{\mu}_m$ of roots of 1, under which $\alpha \in \boldsymbol{\mu}_m$ sends x to αx and y to $\alpha^{-1} y$, and $V/\boldsymbol{\mu}_m = U$. This gives the orbispace structure $\mathcal{C}$ over C, and the map $C_K \to \mathcal{M}$ extends to a map $\mathcal{C} \to \mathcal{M}$.

This gives the flavor of our definition, which we will give below in a general setting.

Here is the lemma we used in the argument:

Lemma 1.5.1 (Purity Lemma, [ℵ-V2])**.** *Let $\mathcal{M}$ be a separated Deligne-Mumford stack, $\mathcal{M} \to \mathbf{M}$ its coarse moduli space. Let X be a separated scheme of dimension 2 satisfying Serre's condition S2. Let $P \subset X$ be a finite subset consisting of closed points, $U = X \smallsetminus P$, and assume that the local fundamental groups of U around the points of P are trivial. Then a morphism $U \to \mathcal{M}$ extends to a morphism $X \to \mathcal{M}$ if and only if the composition $U \to \mathcal{M} \to \mathbf{M}$ extends to a morphism $X \to \mathbf{M}$.*

1.6. **Restrictions on the residue characteristics of the base scheme.** As indicated above, we will need to apply Abhyankar's lemma. This means that we need to assume that no characteristic p appearing divides the order of a stabilizer of a geometric point of $\mathcal{M}$ of characteristic p. This requirement is enough to guarantee that the moduli category described below is an Artin stack with finite diagonal. To get a Deligne-Mumford stack we need a bit more, to ensure that $\overline{\mathcal{M}}_{g,n}(\mathbf{M}, d)$ is a Deligne-Mumford stack: a-priori it is an Artin stack with finite diagonal, but there is a dense open set of primes in $\mathbb{Z}$, depending on g, n, d and $\mathbf{M}$, over which $\overline{\mathcal{M}}_{g,n}(\mathbf{M}, d)$ is a Deligne-Mumford stack. If we denote by $e(M)$ the product of all the "bad" primes listed above, we require that all schemes considered below are schemes over $\operatorname{Spec} \mathbb{Z}[1/e(M)]$. (For simplicity the reader may wish to stick with a base scheme of characteristic 0.)

1.7. **Twisted unpointed nodal curves.** Our first goal is to identify what type of "orbispace curves" we want to work with. It is convenient to start with "unpointed" curves.

Definition 1.7.1. A *twisted nodal curve over S* is a diagram

$$
\begin{array}{c}
\mathcal{C} \\
\downarrow \\
C \\
\downarrow \\
S
\end{array}
$$

Where

1. $\mathcal{C}$ is a Deligne-Mumford stack, with geometrically connected fibers over S, which is étale locally a nodal curve over S;
2. the morphism $\mathcal{C} \to C$ exhibit C as the coarse moduli scheme of $\mathcal{C}$; and
3. $\mathcal{C} \to C$ is isomorphic away from the nodes.

In other words, a twisted nodal curve is a way to put a Deligne-Mumford stack structure $\mathcal{C}$ "over" a nodal curve C. One can give an explicit description of such a structure. For instance, if S is the spectrum of an algebraically closed field k, then locally in the étale topology $\mathcal{C}$ is the stack quotient of $\operatorname{Spec} k[x,y]/(xy)$ by the action of $\boldsymbol{\mu}_m$, where m is prime to the characteristic of k, the parameters x and y are eigenvectors of the action, and the eigenvalues are *primitive m-th roots of 1*.

1.8. **Twisted pointed nodal curves.** Recall that a natural way to get a *pointed* nodal curve C from an unpointed one, is obtained by "separating" some of the nodes and then "ordering" the points above these nodes. These points are disjoint sections of the smooth locus of the curve.

If one "separates" a node on a *twisted* pointed curve, one obtains an object which is a bit more subtle than a section. To see what happens we can look at an étale neighborhood of such a "separated node", which is the stack quotient of $\mathcal{S}pec_S\mathcal{O}_s[x]$ by a faithful action of $\boldsymbol{\mu}_m$ on the variable x. The quotient of the section $\{x=0\}$ in this étale neighborhood is a copy of the classifying stack $\mathcal{B}\boldsymbol{\mu}_m/S$. When these étale neighborhoods are glued together we obtain an *étale gerbe over S*. For our purposes, the reader may think of an étale gerbe $\mathcal{G} \to S$ as a stack, which locally in the étale topology is isomorphic to the classifying stack $\mathcal{B}G/S$ for some finite étale group scheme $G \to S$. (Formally, $\mathcal{G} \to S$ is an étale gerbe when both $\mathcal{G} \to S$ and the diagonal $\mathcal{G} \to \mathcal{G} \times_S \mathcal{G}$ are surjective étale morphisms.) Thus, locally in the étale topology there is a section $S \to \mathcal{G}$, but such sections may not exist globally.

This motivates the following definition:

Definition 1.8.1. A *twisted nodal n-pointed curve over S* is a diagram

$$\begin{array}{ccc} \mathcal{C}_i^s & \subset & \mathcal{C} \\ & \searrow & \downarrow \\ & & C \\ & & \downarrow \\ & & S \end{array}$$

Where

1. $\mathcal{C}$ is a Deligne-Mumford stack which is étale locally a nodal curve over S;
2. $\mathcal{C}_i^s \subset \mathcal{C}$ are disjoint closed substacks in the smooth locus of $\mathcal{C} \to S$;
3. $\mathcal{C}_i^s \to S$ are étale gerbes;
4. the morphism $\mathcal{C} \to C$ exhibit C as the coarse moduli scheme of $\mathcal{C}$; and
5. $\mathcal{C} \to C$ is isomorphic away from the nodes and the $\mathcal{C}_i^s$.

Note that if we let C_i^s be the coarse moduli spaces of $\mathcal{C}_i^s$, then C_i^s embed in C - they are the images of $\mathcal{C}_i^s$, and C becomes a usual nodal pointed curve. We say that $\mathcal{C} \to S$ is a twisted pointed curve of genus g, if $C \to S$ is a pointed curve of genus g.

1.9. **Morphisms of twisted pointed nodal curves.**

Definition 1.9.1. Let $\mathcal{C} \to S$ and $\mathcal{C}' \to S'$ be twisted n-pointed curves. A morphism $F : \mathcal{C} \to \mathcal{C}'$ is a cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & S' \end{array}$$

such that $F^{-1}\mathcal{C}'^s_i = \mathcal{C}_i^s$.

Since twisted pointed curves are stacks rather than schemes, we need to be a bit careful. If $F, F_1 : \mathcal{C} \to \mathcal{C}'$ are morphisms, then we can define a 2-morphism $F \to F_1$ to be an isomorphism of functors. In this way, twisted pointed curves form a 2-category. This may seem to be a problem, since we wish to use them to form a stack, which, by definition, is a category. Here the fact that $\mathcal{C} \to C$ is generically isomorphic comes to rescue: it is easy to see that 2-morphisms are unique when they exist, and replacing morphisms by their equivalence classes we have the following:

Proposition 1.9.2 ([ℵ-V1])**.** *The 2-category of twisted pointed curves is equivalent to a category.*

We call the resulting category *the category of twisted pointed curves*.

1.10. **Stable maps into a stack.** As before, we consider a proper Deligne-Mumford stack $\mathcal{M}$ admitting a projective coarse moduli scheme $\mathbf{M}$. We fix a projective embedding $\mathbf{M} \subset \mathbb{P}^N$.

Definition 1.10.1. A *twisted stable n-pointed map of genus g and degree d over S*

$$(\mathcal{C} \to S, \mathcal{C}_i^s \subset \mathcal{C}, f : \mathcal{C} \to \mathcal{M})$$

consists of a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C} & \to & \mathcal{M} \\
\downarrow & & \downarrow \\
C & \to & \mathbf{M} \\
\downarrow & & \\
S & &
\end{array}
$$

along with n closed substacks $\mathcal{C}_i^s \subset \mathcal{C}$, satisfying:

1. $\mathcal{C} \to C \to S$ along with $\mathcal{C}_i^s$ is a twisted nodal n-pointed curve over S;
2. the morphism $\mathcal{C} \to \mathcal{M}$ is representable; and
3. $(C \to S, C_i^s, f : C \to \mathbf{M})$ is a stable n-pointed map of degree d.

A few remarks are in order.

1. the prefix "twisted" comes to stress the fact that the base curve $\mathcal{C}$ has "extra structure" as a Deligne-Mumford stack. A twisted stable map where $\mathcal{C} \to C$ is an isomorphism is called "untwisted".
2. Twisted stable maps $\mathcal{C} \to \mathcal{M}$ can be defined without invoking the coarse moduli scheme C. For instance, the stability of $C \to \mathbf{M}$ is equivalent to the assertion that $\text{Aut}_{\mathcal{M}}(\mathcal{C}, \mathcal{C}_i^s)$ is finite.
3. The condition that the morphism $\mathcal{C} \to \mathcal{M}$ be representable means that the stack structure on $\mathcal{C}$ is the minimal necessary to ensure the existence of the morphism $\mathcal{C} \to \mathcal{M}$. This should be considered a stability condition, in the sense that it is essential to ensure that the moduli problem be separated.

Now we define morphisms of twisted stable maps:

Definition 1.10.2. A *morphism of twisted stable maps*

$$G : (\mathcal{C} \to S, \mathcal{C}_i^s, f : \mathcal{C} \to \mathcal{M}) \to (\mathcal{C}' \to S', \mathcal{C}'^s_i, f' : \mathcal{C}' \to \mathcal{M})$$

consists of data $G = (F, \alpha)$, where $F : \mathcal{C} \to \mathcal{C}'$ is a morphism of twisted pointed curves, and $\alpha : f \to f' \circ F$ is an isomorphism.

Note that, unlike stable maps into a scheme, a twisted stable map $f : \mathcal{C} \to \mathcal{M}$ may have automorphisms which are trivial on the source $\mathcal{C}$, even when $\mathcal{C} = C$. For example, a family of stable curves over C may have automorphisms fixing C. This is the role of α in the definition.

Again, twisted stable maps naturally form a 2-category. But by the proposition, this 2-category is equivalent to a category. We call this category *the category of twisted stable maps*. In [ℵ-V1] we also give an explicit realization of this category, which is unfortunately a bit technical, in terms of atlases of charts over the coarse curves C. This is in analogy with Mumford's treatment of $\mathbb{Q}$-varieties in [Mu]. Both descriptions are useful when proving results about this category.

It is natural to denote this category $\overline{\mathcal{M}}_{g,n}(\mathcal{M}, d)$, but we find the abundance of $\mathcal{M}$'s a bit confusing. We propose to denote it instead by $\mathcal{K}_{g,n}(\mathcal{M}, d)$.

1.11. **The main result.** There is structural functor $\mathcal{K}_{g,n}(\mathcal{M}, d) \to \mathcal{S}ch$ which associates to a twisted stable map $(\mathcal{C} \to S, \mathcal{C}_i^s, f : \mathcal{C} \to \mathcal{M})$ the base scheme S. With this functor, our main result is:

Theorem 1.11.1. *The category $\mathcal{K}_{g,n}(\mathcal{M}, d)$ forms a Deligne-Mumford stack, admitting a projective coarse moduli scheme $\mathbf{K}_{g,n}(\mathcal{M}, d)$.*

The proof of the theorem is far from easy. After all, many classical moduli problems are solved by stack quotients of appropriate Hilbert schemes, and analogues for Hilbert schemes for stacks are not simple to construct. Our construction builds on the fact that the Kontsevich stack $\mathcal{K}_{g,n}(\mathbf{M}, d)$ is known to be a complete Deligne-Mumford stack, with projective moduli space.

The main steps in our proof are the following.

1. We prove that the diagonal of $\mathcal{K}_{g,n}(\mathcal{M}, d)$ is finite and representable.

2. We show that $\mathcal{K}_{g,n}(\mathcal{M}, d)$ is an algebraic stack, by checking that the condition of [Ar] are verified. Here the hard part is the proof of algebraizability of formal twisted stable maps; this involves a form of Grothendieck's existence theorem for algebraic stacks.

3. We check the valuative criterion for properness for $\mathcal{K}_{g,n}(\mathcal{M}, d)$. For this the main tool is Abhyankar's lemma, together with the purity lemma.

4. We prove boundedness; this is based on showing that $\mathcal{K}_{g,n}(\mathcal{M}, d) \to \mathcal{K}_{g,n}(\mathbf{M}, d)$ has finite fibers, but is a bit more involved. Together with the valuative criterion this implies properness, in particular $\mathbf{K}_{g,n}(\mathcal{M}, d) \to \mathbf{K}_{g,n}(\mathbf{M}, d)$ is finite, thus $\mathbf{K}_{g,n}(\mathcal{M}, d)$ is projective.

1.12. **Balanced maps.** When we first introduced orbispace curves into the picture, the extra structure appeared at a "new" node in the central fiber only in the following way: near a node locally of the form $U : uv = t^r$ we had an orbispace chart, locally of the form $V : xy = t^{r/m}$ with a $\boldsymbol{\mu}_m$ action such that $U = V/\boldsymbol{\mu}_m$. The action was given as follows: $\alpha \in \boldsymbol{\mu}_m$ sends $(x, y) \mapsto (\alpha x, \alpha^{-1} y)$. Note that this is not the most general action of the stabilizer of a node appearing in a twisted stable map: here the eigenvalues of the stabilizer acting on the tangent spaces to the two branches of the node are inverse to each other. A twisted stable map with this property is called *balanced*. We denote the subcategory of balanced twisted stable maps $\mathcal{K}_{g,n}^{\mathrm{bal}}(\mathcal{M}, d)$. We have:

Proposition 1.12.1. *The subcategory $\mathcal{K}_{g,n}^{\mathrm{bal}}(\mathcal{M}, d) \subset \mathcal{K}_{g,n}(\mathcal{M}, d)$ is an open and closed substack. It contains the closure of the locus of twisted stable pointed maps with nonsingular source curve $\mathcal{C}$.*

2. FIBERED SURFACES

2.1. **Fibered surfaces and coarse fibered surfaces.** As our first example, we consider the case $\mathcal{M} = \overline{\mathcal{M}}_{\gamma,\nu}$ of stable ν-pointed curves of genus γ. For simplicity we look at the case $n = 0$ and omit n from the notation.

An untwisted stable map $C \to \overline{\mathcal{M}}_{\gamma,\nu}$ is equivalent to a family $(X \to C, \tau_i : C \to X)$ of stable ν-pointed curves of genus γ over a nodal curve C, such that $\mathrm{Aut}(X \to C, \tau_i)$ is finite. In other words, X is a surface, mapping to C with sections, with stable pointed fibers, satisfying a further stability condition. In general, a twisted stable map $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ is equivalent to a similar family $\mathcal{X} \to \mathcal{C}$, only now $\mathcal{X}$ and $\mathcal{C}$ are not schemes but stacks. Nevertheless, we call the family $\mathcal{X} \to \mathcal{C}$, associated to a twisted stable map, a *fibered surface*. Thus fibered surfaces appear naturally in the boundary of the moduli of untwisted fibered surfaces.

Given a fibered surface $\mathcal{X} \to \mathcal{C}$ one may consider the coarse moduli schemes:

$$
\begin{array}{ccc}
\mathcal{X} & \to & \mathcal{C} \\
\downarrow & & \downarrow \\
X & \to & C.
\end{array}
$$

Now $X \to C$ is not necessarily a family of stable pointed curves; rather, it is locally the *quotient* of such a family by the action of a cyclic group. We call it a *coarse fibered surface*.

2.2. **Comparison with Alexeev's work.** One may ask, what are the singularites of coarse fibered surfaces? And, to what extent can coarse fibered surfaces replace fibered surfaces in the boundary of moduli? In some sense, these questions have already been addressed in the literature. If one restricts attention to *balanced* coarse fibered surfaces, then we show:

Proposition 2.2.1 ([ℵ-V2]). 1. *Let $\mathcal{X} \to \mathcal{C}$ be a balanced fibered surface, $X \to C$ the associated coarse fibered surface. Then X has semi-log-canonical singularities.*

2. *Consider the morphism $X \to \overline{\mathbf{M}}_{\gamma,\nu}$ obtained by composing $X \to C$ with the structural morphism $C \to \overline{\mathbf{M}}_{\gamma,\nu}$. This morphism is a stable map in the sense of Alexeev [Al].*

3. *In characteristic 0, there is a finite morphism from $\mathcal{K}_g^{\mathrm{bal}}(\overline{\mathcal{M}}_{\gamma,\nu}, d)$ to Alexeev's moduli stack of stable maps.*

One can use this to define *the stack of coarse fibered surfaces* as follows. A coarse fibered surface $X \to C \to S$ over a base scheme S gives rise to an Alexeev stable map $X \to \overline{\mathbf{M}}_{\gamma,\nu}$ as well as a Kontsevich stable map $C \to \overline{\mathbf{M}}_{\gamma,\nu}$; and in addition we have the morphism $X \to C$. Since both the Alexeev stacks and Kontsevich stacks are Deligne-Mumford stacks, it is an easy consequence of Grothendieck's theory of the Hilbert scheme that there is a Deligne-Mumford stack A of such triples

$$(X \to \overline{\mathbf{M}}_{\gamma,\nu}, C \to \overline{\mathbf{M}}_{\gamma,\nu}, X \to C),$$

admitting a quasi-projective coarse moduli scheme. There is a morphism $\mathcal{K}_g^{\mathrm{bal}}(\overline{\mathcal{M}}_{\gamma,\nu}, d) \to A$, and the image is the stack of coarse fibered surfaces.

We also show that for suitable choices of the parameters, this morphism is not one-to-one, and is ramified. This means that some of the singularities of Alexeev-type stacks are partially resolved in our stack. In view of this, one can say that the stack of balanced fibered surfaces is, in a sense, a refinement of Alexeev's work on surface stable maps, in the particular case described here.

2.3. **Towards plurifibered varieties.** Our main theorem has a nice recursive feature: the input is a Deligne-Mumford stack with a projective coarse moduli scheme, and the output is of the same nature. It is tempting to apply this feature to higher dimensional varieties. Given a sequence of dominant rational maps

$$X_n \dashrightarrow X_{n-1} \dashrightarrow \cdots \dashrightarrow X_1 \dashrightarrow S$$

of relative dimension 1, one can apply our construction inductively and get, first, a *canonical model*

$$\overline{X_{n\eta}} \to \overline{X_{n-1}{}_\eta} \to \cdots \to \overline{X_{1\eta}} \to \eta$$

over the generic point $\eta \in S$, and, after replacing S by a suitable proper, generically finite and surjective base change, one can get a *stable reduction*

$$\overline{X_n} \to \overline{X_{n-1}} \to \cdots \to \overline{X_1} \to S.$$

We call these structures *stable plurifibered varieties*. Alexeev has suggested to use the minimal model conjecture to define stable varieties (as well as canonical models) in a general situation, without the presence of a "plurifibration" as above. Interestingly, the structure given by the plurifibration allows one to bypass the minimal model program entirely. It would be interesting to compare the singularities of stable plurifibered varieties to the singularities which arise in the minimal model program.

3. Twisted covers and level structures

3.1. **Twisted principal bundles.** We fix a finite group, or a finite étale group-scheme G, and we set $\mathcal{M} = \mathcal{B}G$. We denote the stack $\mathcal{K}_{g,n}(\mathcal{B}G, 0)$ by $\mathcal{B}_{g,n}G$, and $\mathcal{K}_{g,n}^{\mathrm{bal}}(\mathcal{B}G, 0)$ by $\mathcal{B}_{g,n}^{\mathrm{bal}}G$. These we call the stack of *stable twisted G-bundles* and *balanced stable twisted G-bundles*. These names are motivated by the following observation:

There is a nice explicit description of twisted stable maps to $\mathcal{B}G$ in terms of schemes. Say we are working over a base scheme T and $G \to T$ is a finite étale group scheme. Then the associated morphism $T \to \mathcal{B}G$ exhibits T as the universal principal G bundle $\mathcal{P}G \to \mathcal{B}G$. Let $\mathcal{C} \to \mathcal{B}G$ be a twisted stable map. Pulling back the universal principal bundle $\mathcal{P}G \to \mathcal{B}G$ we get an associated *twisted principal G-bundle $P \to \mathcal{C}$*. The fact that $\mathcal{C} \to \mathcal{B}G$ is representable and $\mathcal{P}G$ is a scheme means that P is a scheme. Since $P \to \mathcal{C}$ is étale and $\mathcal{C}$ is nodal, P is a nodal curve, not necessarily connected. Moreover, the action of G on P has no fixed points away from the nodes and the points lying above the marked points of $\mathcal{C}$, and the schematic quotient is C. On the other hand, given such a Galois cover $P \to C$, we can recover $\mathcal{C}$ as the stack quotient of P by G. One easily checks the following:

Proposition 3.1.1. *The category $\mathcal{B}_{g,n}G$ is equivalent to the category of stable n-pointed curves C of genus g, along with nodal G-covers $P \to C$ such that the action is fixed-point free away from the nodes and the marked points.*

The local structure of these stacks is simple. A straightforward calculation of deformation and obstruction spaces, similar to the one in [D-M], gives

Theorem 3.1.2. *The stack $\mathcal{B}_{g,n}G$ is smooth over $\mathbb{Z}[1/|G|]$. Its relative dimension at a given twisted stable map $\mathcal{C} \to \mathcal{B}G$ is $3g - 3 + n - r$, where r is the number of nodal points at which $\mathcal{C}$ is not balanced.*

3.2. **Balanced twisted covers and Galois admissible covers.** In [H-M], Harris and Mumford considered a compactification of the Hurwitz space of simply branched covers of $\mathbb{P}^1$ via *admissible covers*. The construction generalizes to covers of curves of arbitrary genus g with arbitrary ramification type, see [Mo] and [W]. We will now relate these generalized Harris-Mumford stacks of admissible covers with our construction.

The stack $\mathcal{B}_{g,n}^{\mathrm{bal}}G$ contains an open and closed substack parametrizing connected G bundles. The condition for the Galois cover $P \to C$ to be balanced is equivalent to the Harris-Mumford condition for an admissible cover. We have the following:

Proposition 3.2.1. *The subcategory $\mathcal{B}_{g,n}^{\mathrm{adm}}G \subset \mathcal{B}_{g,n}^{\mathrm{bal}}G$ of connected balanced principal bundles is an open and closed substack, which is isomorphic to the stack of Galois admissible covers with Galois group G.*

Note that while the stack described by Harris-Mumford is in general singular, the proposition implies that the stack of Galois admissible covers is smooth. This fact was also observed by Wewers [W]. Remarkably, while Wewers's approach differs from ours, the resulting stacks are the same.

3.3. **Twisted étale covers and admissible covers.** In order to treat admissible covers which are not Galois, we can use the equivalence of categories between d-sheeted étale covers and principal $\mathcal{S}_d$-bundles, where $\mathcal{S}_d$ is the symmetric group on the d letters $\{1, \ldots, d\}$. Given a branched cover $D \to C$ of a smooth base curve of genus g, marked by the n branch points, we take the associated Galois cover $P \to C$ with Galois group $\mathcal{S}_d$. This is an object of $\mathcal{B}_{g,n}^{\mathrm{bal}}\mathcal{S}_d$. Thus, a natural compactification of the moduli of such branched covers is an open and closed substack $Adm_{g,n,d}$ in $\mathcal{B}_{g,n}^{\mathrm{bal}}\mathcal{S}_d$.

Proposition 3.3.1. *Given an object $P \to C$ of $\mathcal{B}_{g,n}^{\mathrm{bal}}\mathcal{S}_d$, the schematic quotient $D = \mathcal{S}_{d-1}\backslash P$ is a d-sheeted cover of C, and $P \to C$ is in $Adm_{g,n,d}$ if and only if $D \to C$ is connected. The branched cover $D \to C$ is admissible in the sense of Harris-Mumford. The stack $Adm_{g,n,d}$ is the normalization of the stack of generalized Harris-Mumford admissible covers.*

Since the stack of Harris-Mumford admissible covers is singular, it is not isomorphic to $Adm_{g,n,d}$. In other words, $Adm_{g,n,d}$ is a minimal desingularization of the Harris-Mumford stack.

There is a definition of $Adm_{g,n,d}$ which does not invoke the principal bundle, only the covering $D \to C$ with extra structure. Consider the following category of *twisted admissible covers:* objects over S consist of

1. a balanced twisted curve $\mathcal{C} \to S$; and
2. a connected finite étale cover $\mathcal{D} \to \mathcal{C}$ of degree n,

satisfying the follwing stability conditions:

1. the morphism $\mathcal{D} \to \mathcal{C}$ is representable;
2. the coarse curve $C \to S$ of $\mathcal{C}$ is stable; and
3. for any geometric point p of $\mathcal{C}$, the action of $\mathrm{Aut}p$ on the fiber $\mathcal{D}_p$ is effective.

We define morphisms by fibered diagrams as usual. We have:

Proposition 3.3.2. *The category of twisted admissible covers is a stack, isomorphic to $Adm_{g,n,d}$.*

3.4. **Principal $\boldsymbol{\mu}_m$-bundles and invertible sheaves.** Stable twisted $\boldsymbol{\mu}_m$-bundles have a natural description via invertible sheaves. We define a category as follows: the objects consist of data $(\mathcal{C}, \mathcal{L}, f)$, where $\mathcal{C}$ is a nodal twisted curve, $\mathcal{L} \to \mathcal{C}$ is an invertible sheaf, and $f : \mathcal{L}^m \to \mathcal{O}_{\mathcal{C}}$ is an isomorphism. We need a stability condition: we require the coarse curve C to be stable, and for each node p on $\mathcal{C}$, we require the action of the stabilizer Stab_p on $\mathcal{L}_p$ to be faithful. We call these objects *stable twisted m-torsion invertible sheaves.* Morphisms of such objects are defined as fibered diagrams. There is a notion of 2-morphisms, making this into a 2-category, but as before it is equivalent to a category.

We claim that these are nothing but Stable twisted $\boldsymbol{\mu}_m$-bundles. Denote by $\mathbf{1}$ the identity section in the total space of the bundle $\mathcal{O}$. Given a stable twisted m-torsion invertible sheaf $(\mathcal{C}, \mathcal{L}, f)$, the inverse image $P = f^{-1}\mathbf{1}$ in the total space of $\mathcal{L}$ is a principal $\boldsymbol{\mu}_m$-bundles, and the stability condition on $(\mathcal{C}, \mathcal{L}, f)$ implies that $P \to \mathcal{C}$ is stable. On the other hand, given a μ_m bundle one has an associated $\mathbb{G}_m$ bundle extending to an invertible sheaf. It is easy to verify that this is an equivalence of categories.

3.5. **Abelian level structures.** There is a remarkable application of the stacks $\mathcal{B}_g G$ to Mumford's moduli of curves with level structures. In this section, we assume that the structure sheaf of the base scheme contains the mth roots of 1, and that we have fixed an isomorphism $\boldsymbol{\mu}_m \simeq \mathbf{Z}/m$. We will construct a smooth complete Deligne–Mumford stack $\mathcal{L}_g^m$, endowed with a finite morphism $\mathcal{L}_g^m \to \overline{\mathcal{M}}_g$, which coincides with the scheme of level m-structures over the stack $\mathcal{M}_g$ of smooth curves of genus g.

The idea is that we can interpret, via Poincaré duality, a level m-structure on a smooth curve C as an element of the cohomology group $H^1\left(C, (\mathbb{Z}/m)^{2g}\right)$; this in turn corresponds to an isomorphism class of $(\mathbb{Z}/m)^{2g}$-bundles. This suggest that we can use our stack $\mathcal{B}_{g,0}(\mathbb{Z}/m)^{2g}$ to define a level m-structure.

Of course there is a problem here: a twisted bundle with group $(\mathbb{Z}/m)^{2g}$ over a fixed smooth curve has $(\mathbb{Z}/m)^{2g}$ as its automorphism group. This is in contrast with the fact that the moduli stack of smooth curves with level structure is representable. This is the same problem that one encounters with the Picard scheme: the stack of line bundles is not representable, because every line bundle has $\mathbb{G}_m$ as automorphisms, and one goes through a process of "removing" this $\mathbb{G}_m$ action and sheafifying (see [FGA], I.B.4, II.C.3, V.1).

This procedure can be carried out in general.

Proposition 3.5.1. *Let $\mathcal{X}$ be a Deligne–Mumford stack. Suppose the automorphism group of every object of $\mathcal{X}$ contains a fixed subgroup G, and that the embedding of this subgroup commutes with base changes. Then there exists a Deligne–Mumford stack $\mathcal{Y}$, equipped with a morphism $\mathcal{X} \to \mathcal{Y}$ which makes $\mathcal{X}$ into an étale gerbe over $\mathcal{Y}$, so that the isomorphism classes of geometric points are the same, but the automorphism group of an object of $\mathcal{Y}$ is the automorphism group of an object of $\mathcal{X}$, divided out by G.*

This fact is certainly known, but we do not know a reference. One can see this using étale presentations, as follows. Take an étale map of finite type $U \to \mathcal{X}$, and set $R = U \times_{\mathcal{X}} U$, so that $R \rightrightarrows U$ is an étale presentation for $\mathcal{X}$. If $\xi \in \mathcal{X}(U)$ is the object corresponding to the morphism $U \to \mathcal{X}$, then R represents the functor $\mathrm{Isom}_{U \times_S U}(\mathrm{pr}_1^* \xi, \mathrm{pr}_2^* \xi)$, where the $\mathrm{pr}_i : U \times_S U \to U$ are the two projections. There is a free action of G on R, leaving the two projections $R \to U$ invariant, defined by composing isomorphisms with the automorphisms associated with G. This allows to define a quotient étale groupoid $R/G \rightrightarrows U$; this is an étale presentation of the stack $\mathcal{Y}$.

Let G be a finite abelian group, and consider $\mathcal{B}_{g,n}^{\mathrm{bal}} G$. Every object of this stack has G in its automorphism group. Applying the proposition, we obtain a stack $\mathcal{B}_{g,n}^{\mathrm{rig}} G$, called the stack of *rigidified* balanced twisted bundles, and a morphism $\mathcal{B}_{g,n}^{\mathrm{bal}} G \to \mathcal{B}_{g,n}^{\mathrm{rig}} G$ as above. Given an *irreducible* curve C and a twisted G bundle $P \to C$, the automorphism group of P over C is equal to G. This means that the restriction $\rho \colon \rho^{-1} \overline{\mathcal{M}}'_{g,n} \to \overline{\mathcal{M}}'_{g,n}$ of the morphism $\rho \colon \mathcal{B}_{g,n}^{\mathrm{rig}} G \to \overline{\mathcal{M}}_{g,n}$ to the open substack $\overline{\mathcal{M}}'_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ of *irreducible* stable curves is representable.

Consider the case $n = 0$, $G = (\mathbb{Z}/m\mathbb{Z})^{2g}$. In this case it is easy to compare the stack $\mathcal{B}_g^{\mathrm{rig}} G$ with Mumford's moduli scheme $\mathcal{M}_g^{(m)}$ of curves with symplectic level-m structure.

Proposition 3.5.2. *There is an open embedding $\mathcal{M}_g^{(m)} \subset \mathcal{B}_g^{\mathrm{rig}}(\mathbb{Z}/m\mathbb{Z})^{2g}$. The closure $\overline{\mathcal{M}}_g^{(m)}$ of $\mathcal{M}_g^{(m)}$ is an open and closed substack, which coincides with the normalization of $\overline{\mathcal{M}}_g$ in $\mathcal{M}_g^{(m)}$.*

The stack $\overline{\mathcal{M}}_g^{(m)}$ can be described directly using symplectic structures. The point is that, for a twisted curve $\mathcal{C}$ underlying an object in $\overline{\mathcal{M}}_g^{(m)}$ over an algebraically closed field, we have $H^1_{\text{ét}}(\mathcal{C}, \mathbb{Z}/m\mathbb{Z}) \simeq (\mathbb{Z}/m\mathbb{Z})^{2g}$, and moreover this group carries a canonical symplectic structure. We can also characterize the "amount of twisting" needed for such a curve: say $\mathcal{C}$ is a *pre-level-m balanced curve* if the stabilized at each separating node is trivial and the stabilizer at a non-separating node is cyclic of order m. We can define a category $\overline{\mathcal{L}}_g^m$ of *twisted curves with level m structure* whose objects are families pre-level-m balanced curves $\pi \colon \mathcal{C} \to S$ along with symplectic isomorphisms $R^1 \pi_* \mathbb{Z}/m\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z})_S^{2g}$, and morphisms given by fibered squares as usual. This forms a stack, and we have

Proposition 3.5.3. *The stack of twisted curves with level m structure is isomorphic to $\overline{\mathcal{M}}_g^{(m)}$.*

3.6. Non-abelian level structures.

We return to an arbitrary finite group G. Consider the stack $\mathcal{B}_g^{\mathrm{adm}} G$ of connected balanced twisted bundles. Objects $P \to C$ over smooth curves correspond to epimorphisms $\pi_1(C) \to G$, once one chooses a base point on C. We can view the stack $\mathcal{B}_g^{\mathrm{adm}} G$ as a stack of stable twisted curves with Teichmüller level structure with group G. In view of the work of Looijenga [L] and Pikaart-De Jong [P-J], it would be interesting to describe the coarse moduli scheme $\mathbf{B}_g^{\mathrm{adm}} G$ and its relation with $\mathcal{B}_g^{\mathrm{adm}} G$. This is the content of work in progress we are conducting with A. Corti and J. de Jong.

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